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Subduction coefficients of Birman–Wenzl algebras and Racah coefficients of the quantum groups $O_q(n)$ and $Sp_q(2m)$: II. Racah coefficients

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Abstract
Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ are derived from subduction coefficients of Birman–Wenzl algebras $C_f(r, q)$ by using the Schur–Weyl–Brauer duality relation between Birman–Wenzl algebras $C_f(r, q)$ with $r = q^{n-1}$ or q^{-2m-1} and the quantum group $O_q(n)$ or $Sp_q(2m)$. It is shown that there are two types of the Racah coefficients according to irreps of $O_q(n)$ or $Sp_q(2m)$ with or without q-deformed trace contraction. The Racah coefficients without q-deformed trace contraction in the irreps involved are n-independent, and are the same as those of quantum groups $U_q(n)$. As examples, Racah coefficients of $O_q(n)$ with q-deformed trace contraction for the resulting irreps $[n_1, n_2, \dot{0}]$ with $n_1 + n_2 \leqslant 2$ are tabulated, which are also Racah coefficients of $Sp_q(2m)$ with substitution $n \to -2m$ and conjugation of the corresponding irreps.

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1. Introduction

Quantized universal enveloping algebras [1–3], often referred to as quantum groups or quantum algebras, are useful algebraic tools in the study of integrable models [4] and conformal field theory [5]. Clebsch–Gordan coefficients of quantum groups can be used to construct universal $\check{R}$ matrices, which are solutions of spectral parameter-free Yang–Baxter equations (YBEs). YBE is an important relation in the construction of exact solutions of these models. While Racah coefficients of quantum groups can be used to obtain new solutions through fusion procedure [6, 7]. The braiding and fusing matrices with relations to Racah coefficients of quantum groups in the rational conformal field theory (RCFT) were shown in [8, 9].

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There are many papers in dealing with Racah coefficients of unitary quantum groups, especially for the simplest case, $SU_q(2)$, by using different methods [10–17]. Some general properties of Racah coefficients for quantum groups were discussed in [17], in which $6j$ symbols for $SU_q(2)$ are used to illustrate the building-up method for calculating the Racah coefficients. However, Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ were never studied before because to construct basis vectors of irreps for them explicitly is not easy. For example, unlike the classical case, there is even no canonical chain for $O_q(n)$. Therefore, to establish a systematic procedure for evaluating these coefficients is difficult in contrast to their classical cases. On the other hand, though there are many methods to derive Racah coefficients for classical Lie algebras, only some special cases for orthogonal and symplectic algebras were considered [18–20] because, generally, outer-multiplicity problem will occur in the couplings and recouplings. The problem will become more difficult for symplectic algebras, in which branching multiplicity will also appear.

In order to overcome these difficulties, we have outlined a procedure in [21] to evaluate Racah coefficients of $O(n)$ and $Sp(2m)$ based on the Schur–Weyl–Brauer duality relation between $O(n)$ or $Sp(2m)$ and Brauer algebra $D_f(n)$. It has been shown that this duality relation can also be extended to the quantum case. The quantum version of this duality relation will enable us to derive the Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ from subduction coefficients (SDCs) of the Birman–Wenzl algebras $C_f(r,q)$ with $r = q^{n-1}$ or q^{-2m-1}. A procedure for evaluating the SDCs of $C_f(r,q)$ has been shown in the previous paper (hereafter referred to as I). In this paper, we will use the results of I to derive Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ with the help of quantum version of the Schur–Weyl–Brauer duality relation and present a general formula for the evaluation.

2. Invariants of the Birman–Wenzl algebras $C_f(r,q)$ and Racah coefficients of $O_q(n)$ and $Sp_q(2m)$

Racah coefficients of $O_q(n)$ or $Sp_q(2m)$ in the unitary form, the so-called U coefficients, can be defined as the elements of a unitary matrix between bases with two different coupling orders of three irreps v_1, v_2, and v_3:

$$|(v_1v_2)v_{12};\, vw\rangle_q = \sum_{v_{23}t_{23}t'} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} |v_1(v_2v_3)v_{23};\, vw\rangle_q^{t_{23}t'} \tag{1}$$

where four multiplicity labels, $t_{12} = 1, 2, \ldots, \{v_1v_2v_{12}\}$, $t = 1, 2, \ldots, \{v_{12}v_3v\}$, $t' = 1, 2, \ldots, \{v_1v_{23}v\}$, and $t_{23} = 1, 2, \ldots, \{v_2v_3v_{23}\}$, are needed, and w labels sub-indices for the irrep v. The Racah coefficients of $O_q(n)$ or $Sp_q(2m)$ in the unitary form obey the following unitarity conditions:

$$\begin{aligned} &\sum_{v_{23}t_{23}t'} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} U_q(v_1v_2vv_3;\, \tilde{v}_{12}v_{23})^{\rho_{12}\rho}_{t_{23}t'} = \delta_{t_{12}\rho_{12}}\delta_{t\rho}\delta_{v_{12}\tilde{v}_{12}} \\ &\sum_{v_{12}t_{12}t} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} U_q(v_1v_2vv_3;\, v_{12}\tilde{v}_{23})^{t_{12}t}_{\rho_{23}\rho'} = \delta_{t_{23}\rho_{23}}\delta_{t'\rho'}\delta_{v_{23}\tilde{v}_{23}}. \end{aligned} \tag{2}$$

The most important fact is that the Birman–Wenzl algebras $C_f(r,q)$ and the quantum groups of B, C, and D types are in Schur–Weyl–Brauer duality. Let $\mathcal{U}_q$ be a quantum group corresponding to a finite-dimensional complex semisimple Lie algebra of type B_n, C_n, or D_n, and let V be the irreducible representation of $\mathcal{U}_q$ corresponding to the fundamental weight. Then the centralizer algebra $\mathcal{C}_f(\mathcal{U}_q) = \mathrm{End}_{\mathcal{U}_q}(V^{\otimes f})$ is isomorphic to a quotient of the Birman–Wenzl algebra $C_f(q^{n-1}, q)$. In the following, we only discuss cases with q being not root of unity. We also need to define a special class of Young diagrams, the so-called n-permissible Young diagrams defined for the quantum groups $O_q(n)$ and $Sp_q(2m)$. A Young diagram $[\lambda]$

is said to be n-permissible if the rational function $Q_\mu(r = q^{n-1}, q) \neq 0$ defined in [22] for all subdiagrams $[\mu] \leqslant [\lambda]$, where the subdiagrams $[\mu]$ can be obtained from $[\lambda]$ by taking away appropriate boxes. In this paper, we only need to discuss the integer n case. It can be verified by using the rational function $Q_\lambda(r, q)$ that a Young diagram $[\lambda]$ is n-permissible iff:

(1) Its first two columns contain at most n boxes for positive n.
(2) It contains at most m columns for $n = -2m$ a negative even integer.
(3) Its first two rows contain at most $2 - n$ boxes for n odd and negative.

When the above three conditions are satisfied, the centralizer algebra $\mathcal{C}_f(\mathcal{U}_q)$ is isomorphic to $\mathcal{C}_f(O_q(n))$ for n positive, to $\mathcal{C}_f(O_q(2-n))$ for n negative and odd, or to $\mathcal{C}_f(Sp_q(2m))$ for $n = -2m < 0$. In the following, we always assume that all irreps to be discussed are n-permissible with $n \leqslant f-1$ for $n > 0$ or $-n \leqslant f-1$ for negative n, which implies that $C_f(q^{n-1}, q)$ to be considered is semisimple.

Under the above conditions, an irrep of $\mathcal{C}_f(O_q(n))$ or $\mathcal{C}_f(Sp_q(2m))$ is also the same irrep of the corresponding quantum group $O_q(n)$ or $Sp_q(2m)$. It should be understood that the labelling schemes of $\mathcal{C}(\mathcal{U}_q)$ and $\mathcal{U}_q$ are different. The former is labelled by its Birman–Wenzl algebra indices, while the latter is labelled by its tensor components. This is the quantum version of the Schur–Weyl–Brauer duality relation.

Similar to Brauer algebra case [21], we can define a Birman–Wenzl algebra invariant as

$$\begin{aligned} \mathcal{W}_{r,q}(\nu_1\nu_2\nu\nu_3;\nu_{12}\nu_{23})^{t_{12}t}_{t_{23}t'} = \sum_{\rho_{12}\rho_{23}\rho} &\left\langle \begin{matrix}[\nu]\\ \rho\end{matrix} \middle| [\nu], \begin{matrix} t\,[\nu_{12}] & [\nu_3] \\ \rho_{12} & \rho_3\end{matrix}\right\rangle_{(r,q)} \\ &\times \left\langle \begin{matrix}[\nu_{12}]\\ \rho_{12}\end{matrix} \middle| [\nu_{12}], \begin{matrix} t_{12}\,[\nu_1] & [\nu_2] \\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)} \left\langle \begin{matrix}[\nu]\\ \rho\end{matrix} \middle| [\nu], \begin{matrix} t'\,[\nu_1] & [\nu_{23}] \\ \rho_1 & \rho_{23}\end{matrix}\right\rangle_{(r,q)} \\ &\times \left\langle \begin{matrix}[\nu_{23}]\\ \rho_{23}\end{matrix} \middle| [\nu_{23}], \begin{matrix} t_{23}\,[\nu_2] & [\nu_3] \\ \rho_2 & \rho_3\end{matrix}\right\rangle_{(r,q)} \end{aligned} \tag{3}$$

where $\left\langle \begin{matrix}[\nu]\\ \rho\end{matrix} \middle| [\nu], \begin{matrix} t\,[\nu_1] & [\nu_2] \\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)}$ is the SDC of the Birman–Wenzl algebras $C_f(r, q)$, and the summation in (3) is carried out under fixed ρ_1, ρ_2, and ρ_3. The invariant (3) only depends on irreps ν_1, ν_2, ν, ν_{12}, ν_{23}, and does not depend on sub-indices.

Using the Schur–Weyl–Brauer duality relation between the Birman–Wenzl algebras $C_f(r, q)$ and the quantum groups $O_q(n)$ or $Sp_q(2m)$, one knows that the Birman–Wenzl algebraic invariants given in (3) are also Racah coefficients in unitary form of $O_q(n)$ when $r = q^{n-1}$, or those of $Sp_q(2m)$ when $r = q^{-2m-1}$. One can thus use (3) to calculate Racah coefficients of B, C, D type quantum groups from SDCs of $C_f(r, q)$ with $r = q^{n-1}$ or q^{-2m-1}, namely,

$$\begin{aligned} U_q(\nu_1\nu_2\nu\nu_3;\nu_{12}\nu_{23})^{t_{12}t}_{t_{23}t'} = \sum_{\rho_{12}\rho_{23}\rho} &\left\langle \begin{matrix}[\nu]\\ \rho\end{matrix} \middle| [\nu], \begin{matrix} t\,[\nu_{12}] & [\nu_3] \\ \rho_{12} & \rho_3\end{matrix}\right\rangle_{(r,q)} \\ &\times \left\langle \begin{matrix}[\nu_{12}]\\ \rho_{12}\end{matrix} \middle| [\nu_{12}], \begin{matrix} t_{12}\,[\nu_1] & [\nu_2] \\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)} \left\langle \begin{matrix}[\nu]\\ \rho\end{matrix} \middle| [\nu], \begin{matrix} t'\,[\nu_1] & [\nu_{23}] \\ \rho_1 & \rho_{23}\end{matrix}\right\rangle_{(r,q)} \\ &\times \left\langle \begin{matrix}[\nu_{23}]\\ \rho_{23}\end{matrix} \middle| [\nu_{23}], \begin{matrix} t_{23}\,[\nu_2] & [\nu_3] \\ \rho_2 & \rho_3\end{matrix}\right\rangle_{(r,q)} \end{aligned} \tag{4}$$

is Racah coefficient in unitary form of $O_q(n)$ for $r = q^{n-1}$, and that of $Sp_q(2m)$ for $q = q^{-2m-1}$ with the corresponding irreps conjugated. Equation (4) will be implemented for evaluating Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ for resulting irreps $[n_1, n_2, \dot{0}]$ of $O_q(n)$ with $n_1 + n_2 \leqslant 2$ by using the SDCs of $C_f(r, q)$ derived in I.

Table 1. Racah coefficients $U_{O_q(n)}(1111;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[0]	[2]	[1^2]
[0]	$-\frac{(q^2-1)r}{(q+r)(qr-1)}$	$\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](q+r)(qr-1)^2}}$	$\sqrt{\frac{(r^2-1)(q^3+r)}{[2](q+r)^2(qr-1)}}$
[2]	$-\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](q+r)(qr-1)^2}}$	$\frac{(r-q)}{[2](qr-1)}$	$-\sqrt{\frac{(q^3+r)(q^3r-1)}{q^2[2]^2(q+r)(qr-1)}}$
[1^2]	$\sqrt{\frac{(r^2-1)(q^3+r)}{[2](q+r)^2(qr-1)}}$	$\sqrt{\frac{(q^3+r)(q^3r-1)}{q^2[2]^2(r+q)(qr-1)}}$	$-\frac{(1+qr)}{[2](q+r)}$

Table 2. Racah coefficients $U_{O_q(n)}(2121;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]	[30]
[0]	$\sqrt{\frac{(q^2-1)^2(1+q^2)r^2}{q(r^2-1)(q+r)(qr-1)}}$	$-\sqrt{\frac{(1+q^2)(r-q)(q^3+r)(1+qr)}{q^3[3](r^2-1)(r+q)}}$	$\sqrt{\frac{q(1+qr)(q^5r-1)}{q^2[3](q+r)(q^3r-1)}}$
[2]	$\sqrt{\frac{(r-q)(qr+1)(q^5r-1)}{q(1+q^2)(r^2-1)(q^3r-1)}}$	$\sqrt{\frac{(q^3+r)(q^5r-1)}{q^4[2][3](r^2-1)}}$	$\sqrt{\frac{q(r-q)}{[3](q^3r-1)}}$
[1^2]	$-\sqrt{\frac{(q^3+r)(qr-1)^2(1+qr)}{q[2](r^2-1)(r+q)(q^3r-1)}}$	$\sqrt{\frac{(r-q)(q^3r+1)^2}{q^3[2][3](r^2-1)(r+q)}}$	$\sqrt{\frac{(q^3+r)(q^5r-1)}{q^2[3](q+r)(q^3r-1)}}$

3. Racah coefficients of $O_q(n)$ and $Sp_q(2m)$

It can be proved that the basic elements, $\{g_i, e_i\}$ defined in I, of the centralizer algebra $\mathcal{C}_f(O_q(n))$ are compatible with reflections $\{g_i \to -g_i,\ e_i \to -e_i\}$, and $n \to -2m$. Hence, there exists an isomorphism between $\mathcal{C}_f(O_q(n))$ and $\mathcal{C}_f(Sp_q(2m))$ by making the maps from $\mathcal{C}_f(O_q(n))$ to $\mathcal{C}_f(Sp_q(2m))$ with reflections $\{g_i \to -g_i,\ e_i \to -e_i\}$, and $n \to -2m$. Its classical counterpart was discussed in [21]. By its q-continuation, it is also valid for the quantum case when q is not a root of unity. In this case an irrep $[\lambda]$ of $\mathcal{C}_f(O_q(-n))$ is the irrep $[\tilde{\lambda}]$ of $\mathcal{C}_f(Sp_q(2m))$ for $n = 2m$, where $[\tilde{\lambda}]$ is the Young diagram conjugate to $[\lambda]$. Such negative dimensionality isomorphism between the orthogonal and symplectic groups was discussed early in [23,24]. While the centralizer algebraic technique asserts its validity [21,22]. Therefore, U coefficients derived from (4) for $O_q(n)$ are also those of $Sp_q(-2m)$ up to a phase factor. Hence, we only need to derive Racah coefficients of $O_q(n)$. In addition, as has been discussed in I, an irrep of the Birman–Wenzl algebra $C_f(r, q)$ is the same irrep of the Hecke algebra $H_f(q)$ when there is no q-deformed trace contraction, namely, $e_i = 0$ for $i = 1, 2, \ldots, f-1$, in this irrep. If q-deformed trace contractions are all zero for every irrep involved in equation (4), it is obvious that the U coefficient in this case is the same as that of quantum group $U_q(n)$ for the same irreps in the couplings and recouplings. In this case, the U coefficients are n-independent. Thus, the tables of Racah coefficients of $U_q(n)$ listed in [25] are also Racah coefficients of $O_q(n)$ or $Sp_q(2m)$. Such type one Racah coefficients for $q = 1$ case has been studied in detail in [25], which, by the q-continuation, applies to generic q cases as well. Similar to the classical case discussed in [21], if the q-deformed trace contractions are non-zero for some irreps involved in the couplings, the corresponding Racah coefficient is called type two coefficients. It is clear that the type two Racah coefficients are generally rank n-dependent. Symmetry properties for the U coefficients of both types are the same as those in the classical case discussed in [21].

Table 3. Racah coefficients $U_{O_q(n)}(1221;\, \nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]	[30]
[1]	$\frac{(q^2-1)r(r-q)}{(q^3r-1)(r^2-1)}$	$-\sqrt{\frac{(r-q)(q^3+r)(qr-1)(q^2r^2-1)}{q^2[3](q^3r-1)(r^2-1)^2}}$	$\sqrt{\frac{(1+q^2)(q^2r^2-1)(qr-1)(q^5r-1)}{q^2[3](q^3r-1)^2(r^2-1)}}$
[21]	$-\sqrt{\frac{(r-q)(q^3+r)(qr-1)(q^2r^2-1)}{q^2[3](q^3r-1)(r^2-1)^2}}$	$\frac{(q^2r^2-1)[2]-r(q^2-1)}{q[3](r^2-1)}$	$\sqrt{\frac{(1+q^2)(r-q)(q^3+r)(q^5r-1)}{q^4[3]^2(q^3r-1)(r^2-1)}}$
[3]	$\sqrt{\frac{(1+q^2)(q^2r^2-1)(qr-1)(q^5r-1)}{q^2[3](q^3r-1)^2(r^2-1)}}$	$\sqrt{\frac{(1+q^2)(r-q)(q^3+r)(q^5r-1)}{q^4[3]^2(q^3r-1)(r^2-1)}}$	$\frac{q(r-q)}{[3](q^3r-1)}$

Table 4. Racah coefficients $U_{O_q(n)}(1^2 1 1^2 1;\, \nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]	[1^3]
[0]	$\sqrt{\frac{(q^2-1)^2(1+q^2)r^2}{q(r^2-1)(q^3+r)(qr-1)}}$	$\sqrt{\frac{[2](1+qr)(q^3r-1)(r-q)}{q^2[3](r^2-1)(qr-1)}}$	$\sqrt{\frac{(q^5+r)(r^2-q^2)}{q[3](q+r)(q^3+r)(qr-1)}}$
[2]	$\sqrt{\frac{(r^2-q^2)(r+q)(q^3r-1)}{(1+q^2)(r^2-1)(qr-1)(q^3+r)}}$	$\sqrt{\frac{(r-q^3)^2(1+qr)}{q^3[2][3](r^2-1)(qr-1)}}$	$-\sqrt{\frac{(q^5+r)(q^3r-1)}{q^2[3](q^3+r)(qr-1)}}$
[1^2]	$-\sqrt{\frac{(r-q)(qr+1)(q^5+r)}{q^2[2](r^2-1)(q^3+r)}}$	$\sqrt{\frac{(q^3r-1)(q^5+r)}{q^4[2][3](r^2-1)}}$	$-\sqrt{\frac{q(qr+1)}{[3](r+q^3)}}$

Table 5. Racah coefficients $U_{O_q(n)}(1 1^2 1^2 1;\, \nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]	[1^3]
[1]	$\frac{(q^2-1)r(qr+1)}{(q^3+r)(r^2-1)}$	$-\sqrt{\frac{(r^2-q^2)(r+q)(1+qr)(q^3r-1)}{q^2[3](r^2-1)^2(q^3+r)}}$	$\sqrt{\frac{[2](q^5+r)(r^2-q^2)(r+q)}{q[3](q^3+r)^2(r^2-1)}}$
[21]	$-\sqrt{\frac{(r^2-q^2)(r+q)(1+qr)(q^3r-1)}{q^2[3](r^2-1)^2(q^3+r)}}$	$\frac{qr(q^2-1)-(r^2-q^2)(1+q^2)}{q^2[3](r^2-1)}$	$\sqrt{\frac{(1+q^2)(q^5+r)(1+qr)(q^3r-1)}{q^4[3]^2(q^3+r)(r^2-1)}}$
[1^3]	$\sqrt{\frac{[2](q^5+r)(r^2-q^2)(r+q)}{q[3](q^3+r)^2(r^2-1)}}$	$\sqrt{\frac{(1+q^2)(q^5+r)(1+qr)(q^3r-1)}{q^4[3]^2(q^3+r)(r^2-1)}}$	$\frac{q(qr+1)}{[3](r+q^3)}$

Table 6. Racah coefficients $U_{O_q(n)}(2 1 1^2 1;\, \nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]
[2]	$-\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$
[1^2]	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	$\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$

By using the results given in I and equation (4), all Racah coefficients of $O_q(n)$ or $Sp_q(2m)$ for the resulting irrep $[n_1, n_2, \dot{0}]$ of $O_q(n)$ with $n_1 + n_2 \leqslant 2$ can be derived. These Racah coefficients of $O_q(n)$ will be listed in tables 1–9, which are all type two coefficients, and are n-dependent. In these tables, one should set $r = q^{n-1}$, and

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \qquad (5)$$

By using the symmetry properties discussed in [21], Racah coefficients with $\nu_1 \leftrightarrow \nu_3$ and $\nu_{12} \leftrightarrow \nu_{23}$ exchanges can be obtained, which are not listed in the tables. On the other hand,

Table 7. Racah coefficients $U_{O_q(n)}(121^21;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]
[1]	$\frac{(q^2-1)r}{(r^2-1)q}$	$\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2(r^2-1)^2}}$
[21]	$-\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2(r^2-1)^2}}$	$\frac{(q^2-1)r}{(r^2-1)q}$

Table 8. Racah coefficients $U_{O(n)}(1^2121;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]
[2]	$-\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$
$[1^2]$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	$\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$

Table 9. Racah coefficients $U_{O_q(n)}(11^221;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]
[1]	$\frac{(q^2-1)r}{(r^2-1)q}$	$-\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2(r^2-1)^2}}$
[21]	$\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2(r^2-1)^2}}$	$\frac{(q^2-1)r}{(r^2-1)q}$

Table 10. Racah coefficients $U_{Sp_q(2m)}(11^21^21;\,\nu_{12}\nu_{23})$.

$\nu_{23}\backslash\nu_{12}$	[1]	[21]	$[1^3]$
[1]	$-\frac{[m+1]}{[2m+1][m-1]}$	$-\sqrt{\frac{[m][2m][2m+4][m+1]}{[3][2m+1]^2[m-1][m+2]}}$	$\sqrt{\frac{[2][m][2m][m-2]}{[3][m-1]^2[2m+1]}}$
[21]	$-\sqrt{\frac{[m][2m][2m+4][m+1]}{[3][2m+1]^2[m-1][m+2]}}$	$\frac{[2][2m]+1}{[3][2m+1]}$	$\sqrt{\frac{[2][m+1][2m+4][m-2]}{[3]^2[m-1][2m+1][m+2]}}$
$[1^3]$	$\sqrt{\frac{[2][m][2m][m-2]}{[3][m-1]^2[2m+1]}}$	$\sqrt{\frac{[2][m+1][2m+4][m-2]}{[3]^2[m-1][2m+1][m+2]}}$	$\frac{[m+1]}{[3][m-1]}$

Racah coefficients of $Sp_q(2m)$ can be obtained by using the relation

$$U_{Sp_q(2m)}(\nu_1\nu_2\nu\nu_3;\,\nu_{12}\nu_{23}) = \eta U_{O_q(n\to-2m)}(\tilde{\nu}_1\tilde{\nu}_2\tilde{\nu}\tilde{\nu}_3;\,\tilde{\nu}_{12}\tilde{\nu}_{23}) \tag{6}$$

where η is an appropriate phase factor, which can be chosen as +1 because the unitarity conditions for Racah coefficients of $O_q(n)$ are satisfied for all n including the negative n case. One can check that minus signs appear under the square root in the expression of the Racah coefficients of $O_q(n)$ in pairs so that we never require to introduce imaginary phase factors for Racah coefficients of $Sp_q(2m)$ after the replacement $n \to -2m$, and can always choose the phase factor η to be +1. As an example, the Racah coefficients of $Sp_q(2m)$ listed in table 10 are obtained from the Racah coefficients of $O_q(n)$ given in table 3 by the replacement $n \to -2m$ and the corresponding irrep conjugations according to the relation shown in (6).

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